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Projective Geometry: Perspectives from
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1 Introduction

It's all in the way you look at it. This is true of many situations, and even more so when regarding the fields of art and mathematics. Some may see art and mathematics as two distinct subjects; how could the rigidity of mathematics be at all related to, or influenced by the fluidity of art? After all mathematics is all about theorems and proofs and right or wrong answers, whereas art is determined by the imagination and interpretation of the artist. Surprisingly amidst these differences art and mathematics *are* connected. During the Italian Renaissance of the 15th century the development of linear perspective in art led to the development of projective geometry as a mathematical discipline. In turn, through further studies within the field of projective geometry mathematics has been able to explain the principles underlying the technique of linear perspective. The following sections will discuss linear perspective and projective geometry both in their own right as well as the unique connection between them.

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2 Linear Perspective, A History

The 15th century Italian Renaissance brought with it many changes and advancements in various fields from art to science and mathematics to religion. One such achievement can be seen in the development of art at the beginning of this Renaissance period, especially the development and use of linear perspective. The use of linear perspective in art arose out of the desire to depict on a flat surface what is perceived by the eye.

Linear perspective differed from previous medieval art of the Middle Ages which had a flat playing card look to it. An example of this flat medieval art can be seen below in the painting The Nativity by Duccio (1255-1318). Both the overlapping of the on-lookers as well as the lack of foreground contributes to the flat look of the image. Furthermore there is a lack of proportionality of characters in the image, notice the size of Mary as compared to the men in the foreground.



Figure 1: The Nativity by Duccio. An example of Medieval art. [6]

At the onset of the 14th century a transition began from this flat look to images that more convincingly portrayed depth through the use of linear perspective. In The Lamentation and The Legend of St. Francis, both by Giotto di Bondone (1267-1337), such differences in the portrayal of depth can clearly be seen (figure 2 below). The Lamentation displays some of the

common qualities of medieval art, both an overlapping of the characters and a background which is drawn flat. But in The Legend the evolution toward linear perspective can be seen. The room is drawn such that depth is portrayed as it would be seen; this is achieved by the convergence of the ceiling lines, and the addition of the arches along the ceiling. However the characters in the image still have the overlapping quality as was used in medieval art.



Figure 2: The Lamentation (left) [4] and The Legend of Saint Francis (right) [5] both by Giotto. A comparison between Medieval art and the transition to Renaissance perspective.

Some guidelines used by Giotto in his work include: lines and planes above eye level of the viewer should appear to slope downward and move away from the viewer, and those below should appear to slope upward and away from the viewer. Similarly, lines to the left of the viewer should tend to the right, while lines to the right of viewer tend to the left [9]. Figure 3 below gives an analysis of The Legend that shows the parallel lines of the ceiling and arches abide by these principles, and tend toward a central vertical line.

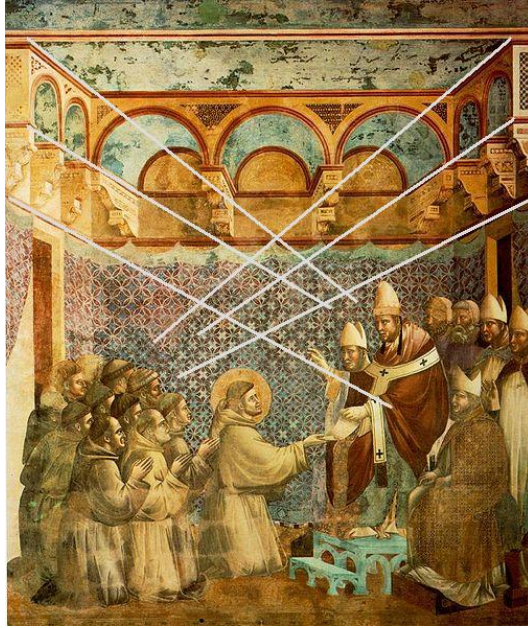


Figure 3: An analysis of converging lines in The Legend of St. Francis

The concept of linear perspective was becoming common among artists, but there were no consistent techniques. Each artist implemented his own personal technique to portray depth and space as he saw best. But around the turn of the 15th century came Filippo Brunelleschi (1337-1446) who is credited with inventing linear perspective. Through two ‘demonstration paintings’, of the Baptistery of St. John and the Florence government palace (c.1413), Brunelleschi achieved what may be considered ‘scientific accuracy’ in his paintings [9]. Unique aspects of Brunelleschi’s technique were not only that he used the building themselves as physical subjects, but also that there was a specific technique used for viewing the painting, to heighten the illusion of perspective.

Despite the effectiveness and convincing portrayal of reality Brunelleschi attained through these demonstration paintings, the practicality of the art was not desirable at this time in history. First of all such scenes were not popular subjects, and furthermore the requirement of the artists to literally view the subject was not very practical. Thus the artists that followed Brunelleschi while willing to adopt the concept of linear perspective, continued to use their own subjective techniques to create perspective. One such artist to adopt perspective and

use it to great effect was Masaccio (1401-1428). In the Holy Trinity (figure 4), Masaccio used a single vanishing point to portray perspective and space. Martin Kemp gives an analysis of Masaccio's Holy Trinity that displays this technique of converging lines to portray depth. It can be seen in figure 4 that the lines of the tiles in the ceiling vault converge to the point *V* just below the stair. As we have seen in other paintings, the technique of converging lines was common, but never as precisely executed as in Masaccio's Holy Trinity.

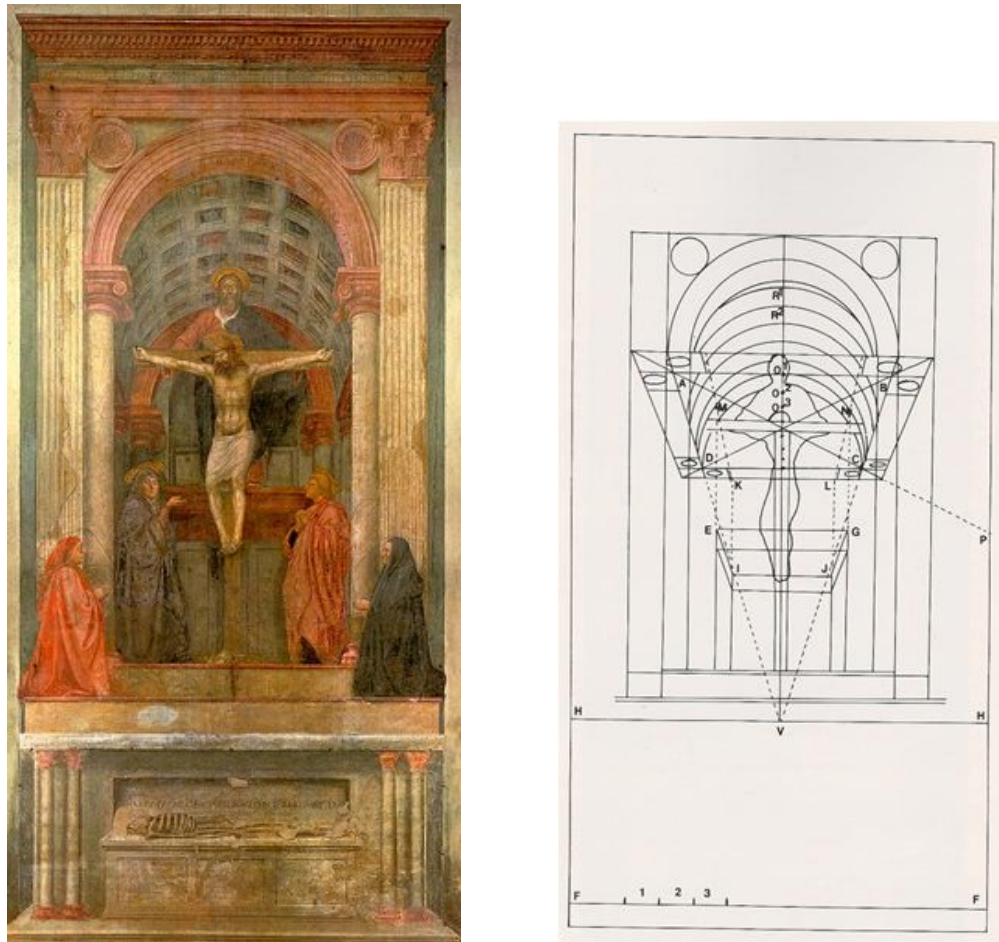


Figure 4: The Holy Trinity by Masaccio (left) [11]. Analysis of perspective in Holy Trinity by Kemp [9]

With increased use of linear perspective developing in art, the technique began to be passed down from master to pupil. However, there remained very few formal writings on the subject and its techniques until the treatise “Della Pittura (On Painting)” by Leon Battista Alberti in 1435. In this treatise Alberti outlined and described many of the proper and

effective techniques for painting in perspective. Alberti emphasized three topics as the basis of the treatise: visible reality, mathematics, and the means and aim of humanist painting [2]. Alberti himself admits he is not a mathematician and his words should be “interpreted solely as those of a painter” [2]. So in the words of a painter he defines and describes terms such as point, line, and plane with an emphasis on how they are seen and used in art. For example in defining a plane Alberti writes:

The plane is that certain external part of a body which is known not by its depth but only by its length and breadth and by its quality. Some qualities remain permanently on the plane in such a manner that they cannot be changed without altering the plane itself. Other qualities are such that, due to visual effects, they seem to change to the observer even though the plane remains the same [2].

Here the visual effects Alberti refers to apply specifically to art whereas the other descriptions are also applicable to mathematics. Such visual effects that seem to alter the appearance of the plane are those of light and color, but neither light nor color actually changes the quality of the plane itself.

Once definitions have been addressed Alberti can begin to address a key aspect of his linear perspective technique, visual rays. Visual rays are simply lines of sight from the eye. Alberti uses the concept of visual rays as a tool to measure the plane and “carry the form of the thing seen to the sense” [2]. Among these rays there are three distinctions to be made: extreme rays are those that strike the outline of the plane and measure its quantity, median rays which quantify color and light, and the centric ray such that where this one touches the plane it makes equal right angles all around it [2]. An example of this can be thought of when you are viewing a cube. The extreme rays are those that ‘see’ the corners and edges of the cube, the median rays are those that quantify any shadows on or near the cube, and the centric ray is the exact point at which you are looking. With these rays, Alberti then defines the visual pyramid “The base of this pyramid is a plane which is seen. The sides of

the pyramid are those rays which I have called extreme. The cuspid, that is the point of the pyramid, is located within the eye” [2].

Of the different rays that make up the visual pyramid the centric ray is most important being the “last to abandon the thing seen,” and as such when the distance and position of this ray are changed the plane will appear altered. Specifically changing the position of the centric ray changes the focal point of the image. The median rays are also capable of an apparent altering of the plane by the way light is received. The extreme rays are those that account for form, creating an outline of the subject which is to be drawn. While this terminology is no longer used, the basic concept remains central to the technique of linear perspective. Today this concept is known as the cone of vision. An example of the modern cone of vision is pictured below in figure 5.

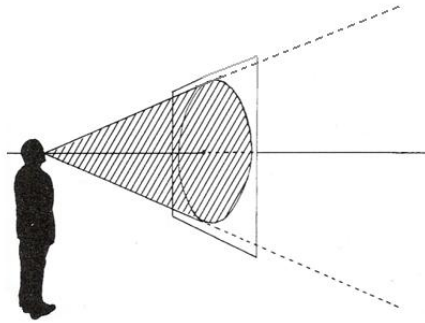


Figure 5: The cone of vision with base in a plane [7].

The cone of vision, and similarly Alberti’s visual pyramid, both have the apex at the eye and a base that is unfixed, but when a plane intersects this cone, the base then lies within that plane [7]. Alberti uses his visual pyramid to assess different planes of a whole scene, such as pavement and walls, each plane representing a different base of the visual pyramid. Together the various planes create an image that is like a cross-section of the pyramid [2]. To further employ this interpretation of the visual pyramid, Alberti cites a mathematical axiom regarding proportional triangles. That is: “if a straight line cuts two sides of a triangle, and if this line which forms a triangle is parallel to a side of the first and greater triangle,

certainly this lesser triangle will be proportional to the greater” [2]. Proportional triangles are said to be those whose sides and angles contain a ratio to each other. To use this axiom Alberti creates triangles with base in the subject plane and visual rays as the sides. Then proportional triangles will be created where the picture plane intersects these rays (assuming the picture plane is parallel to the subject plane). Because of this mathematical principle Alberti is able to preserve proportionality in his art work.

Let’s briefly explore proportional triangles and the consequences they have here for Alberti and his work. Suppose we have the situation as shown in figure 6 below where the artist is positioned a distance x away from a picture plane, and a distance y away from the subject plane. If there are objects in the subject plane of heights A and B that are to be drawn on the picture plane Alberti’s axiom on proportional triangles says that if the picture plane cuts the ray from the artist to the subject parallel to the subject plane, then this lesser triangle will be proportional to the greater. From this proportionality, ratios of sides of these triangles may be created such that $\frac{x}{a} = \frac{y}{A}$ and $\frac{x}{b} = \frac{y}{B}$. From here by cross multiplication we get the following equalities: $\frac{a}{A} = \frac{x}{y} = \frac{b}{B}$. We may then solve for a and b to determine appropriate scaling in order to preserve proportionality, specifically $a = \frac{x}{y}A$ and $b = \frac{x}{y}B$, and the scaling factor is $\frac{x}{y}$.

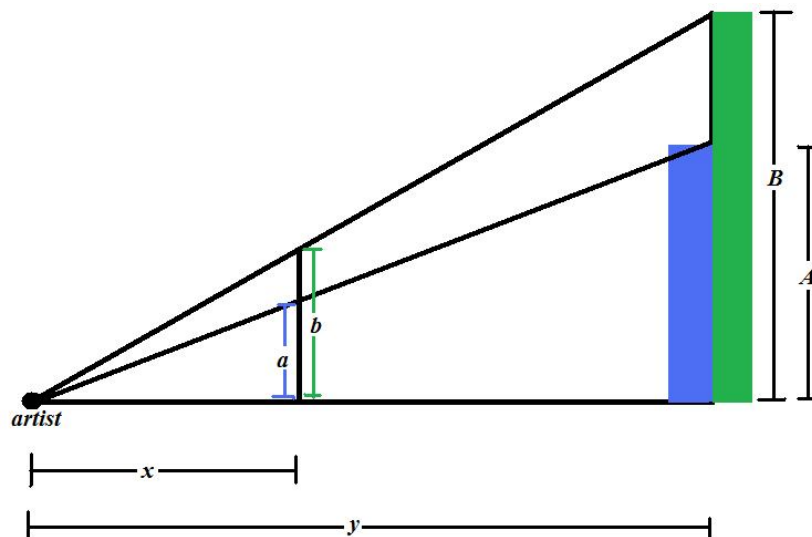


Figure 6: An example of how proportional triangles and ratios may be used to preserve proportionality.

Another thing to note from these ratios is what occurs when the object gets farther away from the artist. If we begin as before with the artist at a fixed point, the picture plane a distance x away and the object with height A at a distance y , solving as before yields $a = \frac{x}{y}A$, which should be the height of the object in the picture plane. But now suppose we take the object and move it farther away to a distance of y' , where $y < y'$. If $y < y'$ then $\frac{x}{y'} < \frac{x}{y}$ and $a' = \frac{x}{y'}A < a = \frac{x}{y}A$. Figure 7 below shows how the triangle changes when the object is moved farther way. It is evident then that the object will appear smaller on the picture plane as it gets farther away from the artist.

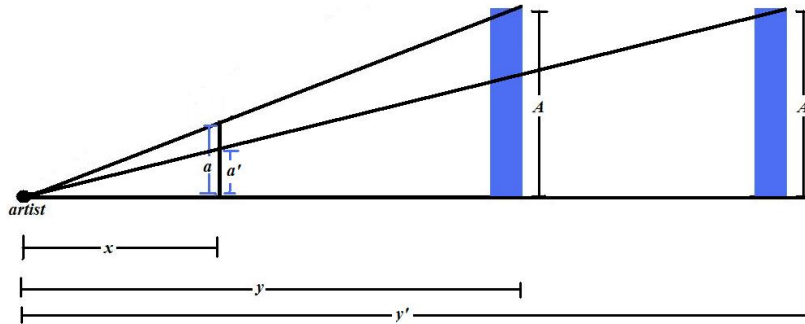


Figure 7: An example of how distance affects size of an object in the picture plane.

The above prescriptions are given in the mathematical portion of Alberti's treatise while further discussions in the treatise deal with the artistic aspects of painting. Once the mathematics assist in creating the form of the painting, further aspects such as light and color as well as emotion must be considered to fully create what Alberti believes to be genuine art.

3 Linear Perspective Today

Today linear perspective is taught in many art classes as a basic drawing technique. In these classes there are three types of perspective that are emphasized, one-point, two-point and three-point perspective. Each type of perspective uses lines to indicate depth and an illusion

of convergence. The points that are referred to in their titles are the number of vanishing points to which certain parallel lines converge. We will briefly explore these three types of perspective and give examples of each. For the cases of one and two-point perspective the base of the object is assumed to be parallel to the ground plane.

One-point perspective is defined by a single vanishing point that lies on the horizon line. The horizon line is a horizontal line that runs across the picture plane at eye level of the viewer. A common use of one-point perspective as shown in figure 8 below is when drawing the interior of a room. Here the horizon line is at eye level of the artist or viewer and the vanishing point on that line is such that all parallel lines that run away from the artist converge to that vanishing point. If the drawing plane is assumed vertical, then all lines or faces of objects parallel to the drawing plane remain parallel in the drawing. For example a picture hanging on the wall opposite the artist will not be distorted. The second image in figure 8 is that of a cube drawn in perspective, with parallel lines receding away from the viewer and converging at the center of the cube.

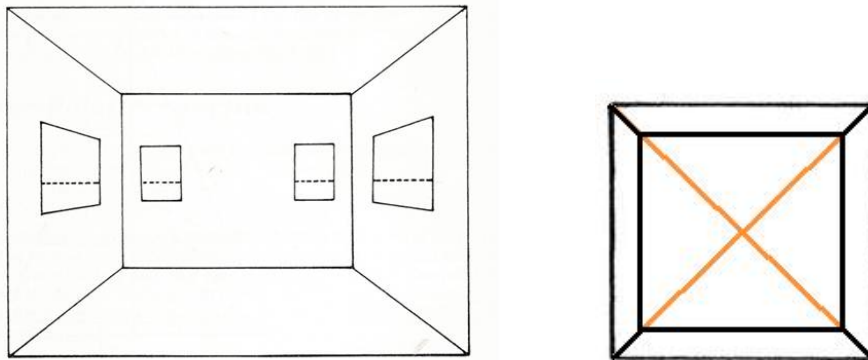


Figure 8: Examples of one-point perspective. The interior of a room (left) [7] and a cube drawn in perspective with diagonals drawn in orange converging to the vanishing point at the center (right).

Two-point perspective is similar to one-point perspective in that there is a horizon line, except on this horizon line there will be at least two vanishing points. Such examples of two-point perspective arise when an object is viewed on an angle, so that no face or lines are parallel to the drawing plane. In some cases if different objects appear at different angles

there will be two vanishing points corresponding to each object. This can be seen by the cubes in the first image figure 9, where the dashed lines will converge at different vanishing points for each object. A common use of this type of perspective is seen in the second image of figure 9; that is standing at an intersection looking at the corner of a building where the sides of the building converge down the streets.

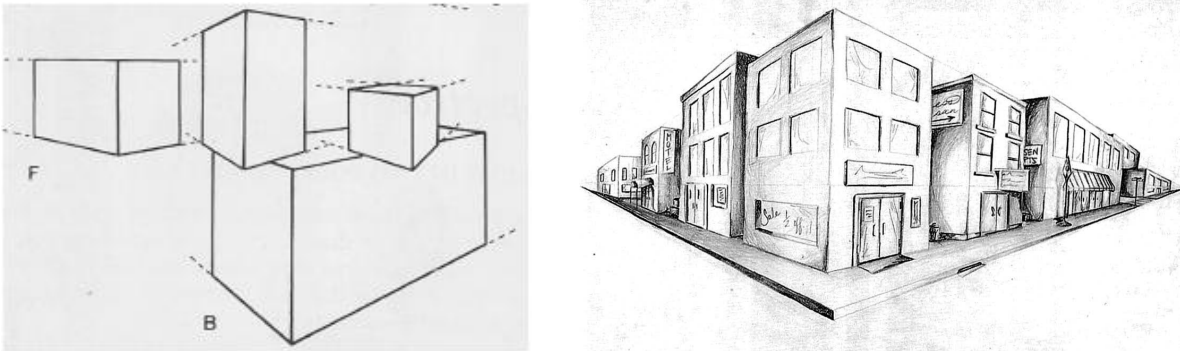


Figure 9: Examples of two-point perspective. Multiple cubes with two vanishing points for each (left) [7] and a street scene in two-point perspective [13].

Three-point perspective, as with two-point perspective, has two or more vanishing points on the horizon line, but also an additional vanishing point either above or below the horizon line. This type of perspective can be thought of as a birds-eye view or a bugs-eye view. Here not only do the sides of the object appear to recede away from the viewer, but also the bottom or top respectively. In three-point perspective it is not necessarily the case that the base of the object will be parallel to the ground plane. The first images in figure 10 show that the base of the object may be tilted away from the ground plane. Three-point perspective is commonly used when drawing a skyscraper, either from the air so that the vertical lines appear to converge below the building, or from below so that the vertical lines appear to converge above. The second image of figure 10 shows the image when looking up at a skyscraper so that the vertical lines converge to a point (labeled P2) above the object.

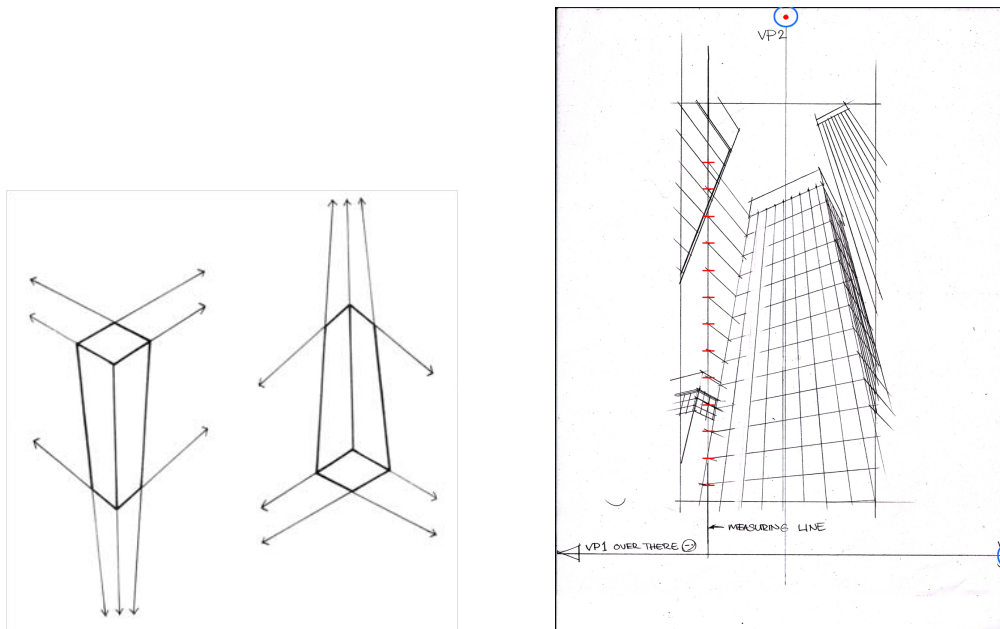


Figure 10: Examples of three-point perspective. A rectangular block with base tilted away from the ground plane (left) [7] and a skyscraper as seen from a bugs-eye view (right) [1].

This technique is very prescriptive in how it creates linear perspective, the vanishing points are usually determined first, and then the drawing is created in relation to them. However the same results can be obtained when using Alberti's technique of the visual pyramid. Using Alberti's technique of the visual pyramid we will take the visual rays from the artist to the object and note where they intersect the picture plane, thus creating an image in the picture plane. We will now illustrate how this technique from Alberti corresponds to the modern principles of linear perspective and vanishing points. The images that follow were created by using the mathematical software Maple.

We begin in 3-space (defined by axes x, y, z) with a cube, centered at the origin, an eye-point, and a picture plane as seen in figure 11.

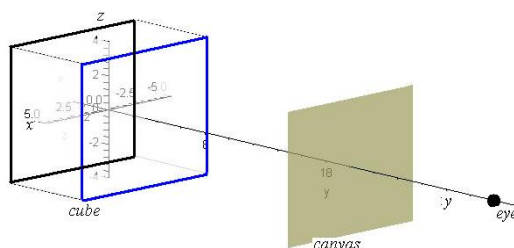


Figure 11: A cube centered at the origin, a picture plane and an eye-point in 3-space. [10].

For these examples let the cube remain fixed at the origin, and allow the eye point and the picture plane to move in relation to one another so that the picture plane is always between the cube and the eye point and perpendicular to the line that connects the eye point to the origin.

For one-point perspective let the eye point lie on the y - axis some distance away from the origin. Each face of the cube that is parallel to the picture plane can be thought of as a different base of the visual pyramid. Now let us connect each corner of the cube to the eye point with a line; these extreme rays of the pyramids (as Alberti would call them) will intersect the picture plane as seen in figure 12 below. The blue lines coming from the corners of the front face, and the black lines from the corners on the back face. The points where these lines intersect the picture plane can then be connected to create the image of the cube. Notice since both the front face and the back face are parallel to the picture plane their boundary lines are not distorted, they remain equal on each face. However since the back face, is farther away from the eye point, it will appear smaller. Thus the image on the picture plane, seen here as the second image of figure 12, appears as two concentric squares, with diagonals converging toward the center of the two squares. This matches the one-point perspective prescription from above.

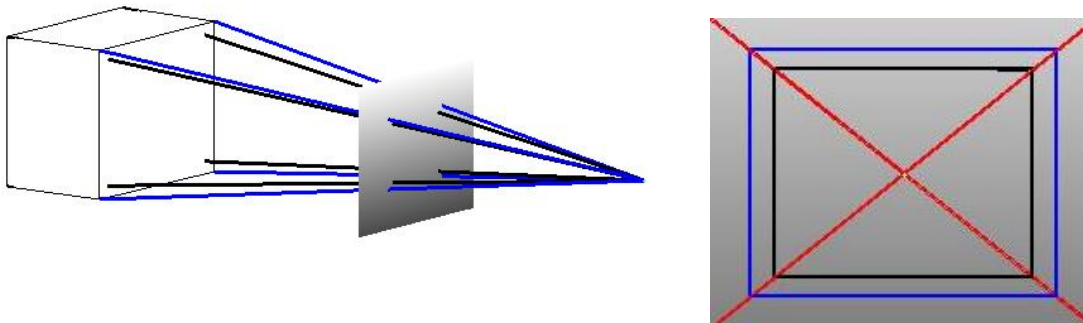


Figure 12: The set up for one-point perspective with extreme rays drawn from the corners of the cube to the eye point on the y - axis (left). The resulting image constructed from the intersections of the rays with the picture plane (right). [10]

To achieve two-point perspective let us move the eye point, away from the $y - axis$ but still in the $xy - plane$, and let us move the picture plane accordingly. Again we connect the corners of the square to the eye point, and take note of where these connecting lines intersect the picture plane. The set up is shown in the first image of figure 13. Here the bases of the visual pyramids are not parallel to the picture plane; one base remains on the front face as in one-point perspective, but the other is now on the right side face of the cube. Both are skewed so that one side of the base is farther away from the eye point which makes that edge of the cube appear shorter (as we would expect from our discussion on ratios). The image of the cube can now be created from the intersection points on the picture plane as before and the result is shown in the second image of figure 13. It may not be as clear this time, but if we extend the edges of the top and bottom faces (the green lines), they will converge at a point to the left of the cube, and the two edges of the side faces (the red lines), they will converge at a point to the right. Now it is clear that this resulting image again matches the two-point perspective as described above.

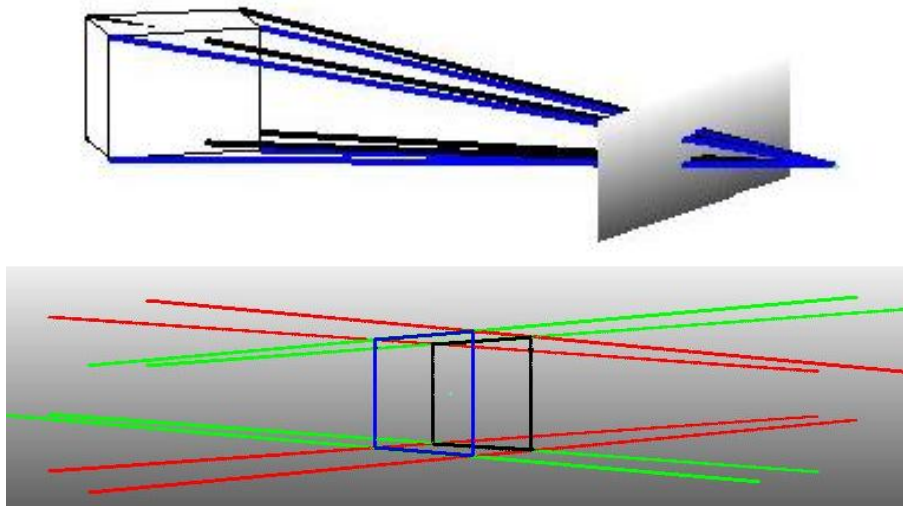


Figure 13: The set up for two-point perspective with the extreme rays drawn from the corners of the cube to the eye-point shifted away from the $y - axis$ and the picture plane moved accordingly (top). The resulting image on the picture plane constructed from the intersections of the rays (bottom). [10]

We can extend this idea to obtain a three-point perspective image. From our two-point

perspective position, let us move the eye point up in the positive z direction, out of the $xy - plane$, and then move the picture plane accordingly. The first image in figure 14 shows the initial set up with extreme rays connecting the corners of the cube to the eye-point. There are now three pyramids with bases on the front, right, and top faces of the cube. As in the two-point perspective case some edges again are farther away from the eye point, and thus the corresponding lines in the picture plane will be shortened according to the appropriate ratios. By connecting the corners of the square to the eye point, and creating the image on the picture plane as before we obtain a three-point perspective image, shown in the second image of figure 14. This image is similar to that of the two-point perspective, where the top and sides converge, but now also the vertical (yellow) lines of the cube converge downward. Once more we see this is consistent with the linear perspective as described above.

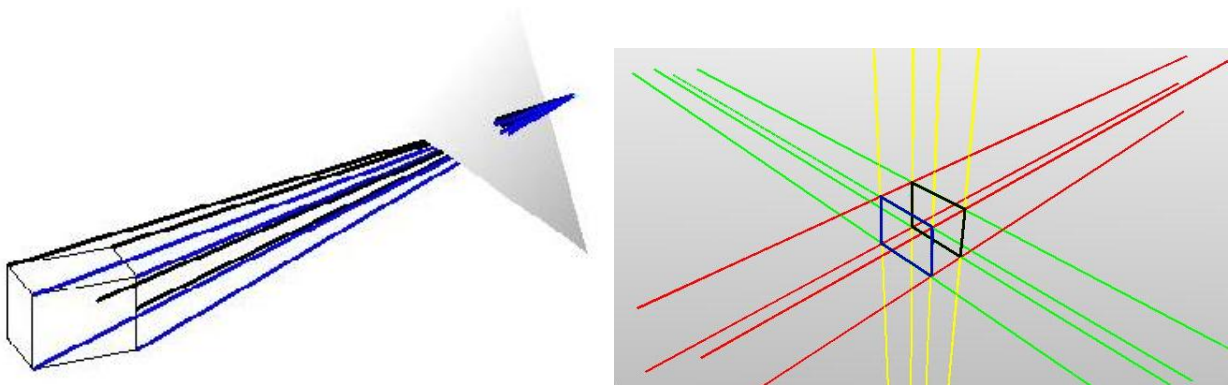


Figure 14: The set up for three-point perspective with extreme rays drawn from the corners of the cube to the eye-point shifted away from the $y - axis$ and up in the positive z direction and the picture plane moved accordingly (top). The resulting image constructed from the intersections of the rays with the picture plane (bottom) [10]

From these examples we see that the classical technique of Alberti and his visual pyramid, and the current technique taught in art classes today, when employed properly yield the same perspective images.

4 Projective Geometry, A History

What is known today as the field of projective geometry arose from this practice of linear perspective in art. Mathematicians and artists alike were interested in the principles that governed this artistic technique and therefore began to investigate the underlying reasons for these principles. In particular artists were concerned with lines of sight and parallel lines that converged and intersected at a vanishing point, while mathematicians were interested in determining a system that would encapsulate these principles. As a result of this shared interest between artists and mathematicians, many of the early theorems from projective geometry are specifically concerned with perspective.

The first mathematician credited with creating and contributing to projective geometry is Girard Desargues (1591-1661). Desargues spent much time focused on the mathematics and geometry of perspective. One of his most well known papers appropriately titled, *Perspective*, discusses his studies and findings regarding both the art and mathematics of perspective.

As part of his artistic assessment Desargues begins *Perspective* by describing a technique for drawing in perspective “without using any third point,... which lies outside the picture field” [8]. As compared to the common techniques used today, Desargues method is quite different. Today most perspective techniques begin by determining the vanishing points and then creating the images in relation to the established vanishing points. Here however Desargues outlines a technique for creating an image without determining the vanishing points first. And what is more, if there were to be a vanishing point for the image, it would not even lie within the picture plane.

This technique essentially relies on measurements of the subject and scale constructions within the picture plane. These scales are constructed from the measurements and are used in such a way as to preserve distance and dimension in the image. Again we see the concern for proportionality, as with Alberti. Here rather than Alberti’s technique of proportional triangles, Desargues uses carefully constructed scales to maintain proportionality in his image.

Once the scales have been established the artist can simply use a compass and straight edge to depict the desired subject appropriately in perspective. In figure 15 below we see Desargues' original work, an image of a birdcage drawn by this technique. Note also the vanishing point lying well off the page.

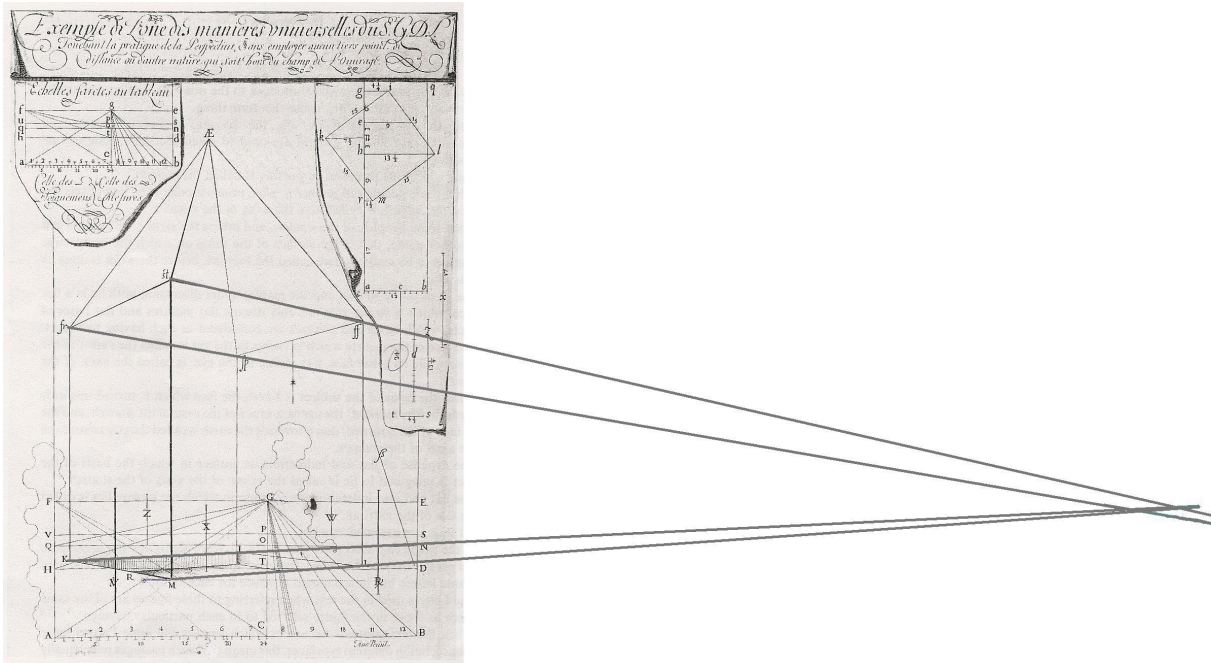


Figure 15: The document of Desargues' linear perspective construction with the vanishing point shown off the page [8].

Following this perspective construction Desargues continues with three geometrical propositions that follow the theme of perspective. The culmination of these three propositions is known today as Desargues' Theorem. This theorem shows how two triangles are related when drawn in perspective. The theorem states: If the three straight lines joining the corresponding vertices of two triangles all meet in a point, then the three intersections of pairs of corresponding sides lie on a straight line [14].

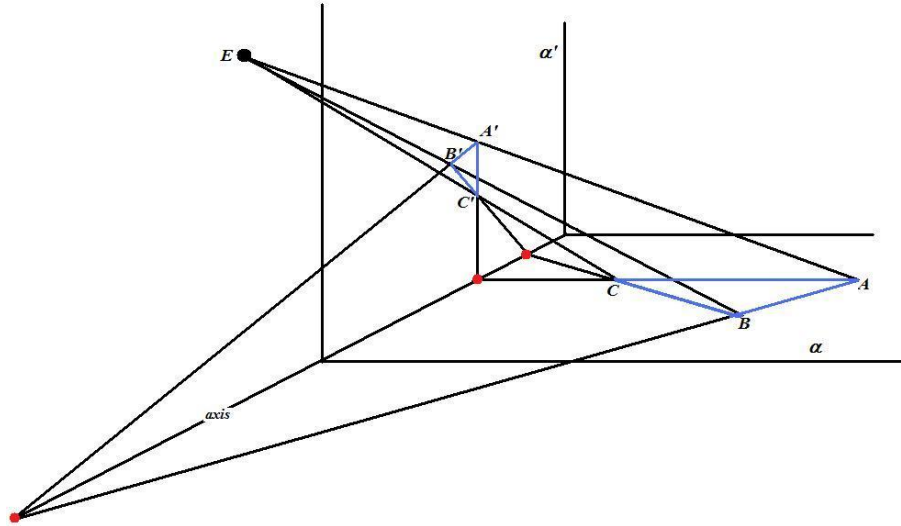


Figure 16: An illustration of Desargues' Theorem [12].

It is clear the connection that can be made between this theorem and linear perspective. As the image suggests this is another example of Alberti's visual pyramid technique. Here triangle ABC lies within the ground plane, the base of the pyramid, and triangle $A'B'C'$ lies within the picture plane, and the point at which the lines (visual rays) joining corresponding vertices meet E , is the eye-point of the artist, the apex of the visual pyramid.

Furthermore, as would be expected, there is a connection to be made between this theorem and modern projective geometry. Consider the intersection of the pair of corresponding sides AB and $A'B'$. From figure 16 we see that they both lie in the same plane, namely the side of the pyramid they form. If these lines intersect in the plane in which they lie, this is the point of intersection we are guaranteed by the theorem. On the other hand, if these two lines are parallel in the plane in which they lie, they will not intersect in that plane. However the theorem guarantees these lines will intersect; thus if this theorem is to remain consistent for all triangles, Desargues must introduce a point at which these two parallel lines will intersect, he chooses 'a point at infinity'. We will see in the following section further developments in projective geometry and how the original notions from Desargues are made formal with the projective geometry of today.

5 Developing Projective Geometry

Since the time of Desargues, projective geometry has expanded to become a unique field in mathematics. It is now most commonly thought of as a space in which geometrical objects live and act. However this modern geometrical space can be developed in the same manner as was classical projective geometry from the notions of linear perspective and art. Let us explore how linear perspective and art lead to this development of projective space.

First let us claim that on any line of sight there is only one point that is seen. The object that is nearest will contain that point which is seen. For example imagine looking at a box from the front. The front face is closest to you on your line of sight so the place where your line of sight intersects the front face represents the point corresponding to that line of sight. So for any line of sight you choose, the result is that you see a single point corresponding to that line of sight. To formalize this idea mathematically we define an equivalence relation \sim such that $(a, b, c) \sim (\lambda a, \lambda b, \lambda c)$ and we denote $\overline{(a, b, c)} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) = \lambda(a, b, c) \text{ for } \lambda \in \mathbb{R}\}$. That is to say we let any other point, $(\lambda a, \lambda b, \lambda c)$, lying on the same line as (a, b, c) be equivalent \sim to (a, b, c) and denote the set of all such points as $\overline{(a, b, c)}$. The points (a, b, c) are the points of Euclidean space, while the points $\overline{(a, b, c)}$ are the points of projective space.

With this premise let us begin our construction of projective space. Imagine you are standing at the origin and there is a canvas in the $y = 1$ plane in Euclidean 3-space. You can look in any direction you would like and you will see one point lying on your line of sight. If you extend your line of sight from you to the object infinitely in both directions, through the origin, the line will do one of two things: (1) this line will intersect the $y = 1$ plane, or (2) this line is parallel to the $y = 1$ plane and lies in the xz -plane (this is the same as the $y = 0$ plane, and points in this plane have the form $(a, 0, b)$). Two examples of such lines are shown in figure 17 below. If the line intersects the $y = 1$ plane this point of intersection is the point in projective space corresponding to that line through the origin in

Euclidean 3-space. We may choose the Euclidean point of intersection $(a, 1, b)$ to represent the projective point $\overline{(a, 1, b)}$. However if the line is parallel to the $y = 1$ plane, we must introduce a new point in projective space to correspond to that line. We introduce a point at infinity which corresponds to the particular direction of this line. So any line of sight we choose will have a corresponding point in projective space that belongs to one of two sets: either the set of points in the $y = 1$ plane, a copy of \mathbb{R}^2 , or the set of points at infinity. The collection of all elements in these two sets is precisely the definition of projective space; $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{\text{points at } \infty\}$, where projective space denoted $\mathbb{P}^2(\mathbb{R})$, is the set of all points either in the set of (x, y) pairs from \mathbb{R}^2 or a point at infinity (∞). From this example we see that a line through the origin in Euclidean 3-space corresponds to a point in projective space.

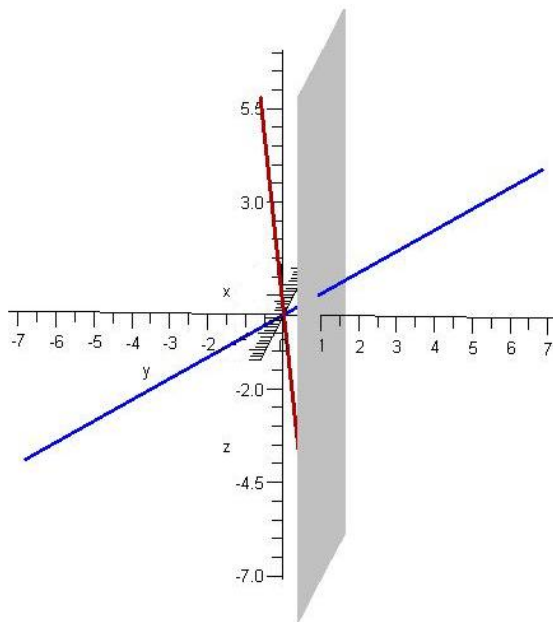


Figure 17: An example of two lines through the origin in 3-space: one intersecting the $y = 1$ plane (blue) and the other parallel to the $y = 1$ plane (red).

With this definition of projective space let us next explore lines in projective space. In Euclidean geometry a line is typically defined by $y = mx + b$ in an xy -plane. Here in keeping with our artist and canvas set up we will define our Euclidean lines to be in the $y = 1$ plane. As such we will define a line with the equation $Z = mX + n$, or in an equivalent

form $aX + bZ + c = 0$. In projective space however the line $aX + bZ + c = 0$ is not well defined. Going back to our equivalence relation \sim if the point (X_0, Z_0) satisfies $aX + bZ + c = 0$ then by \sim so should the point $(2X_0, 2Z_0)$. However upon substitution we see that

$$a(2X_0) + b(2Z_0) + c = aX_0 + bZ_0 + (aX_0 + bZ_0 + c)$$

where, by hypothesis, $(aX_0 + bZ_0 + c) = 0$ but it is not necessarily true that $aX_0 + bZ_0 = 0$.

If we think of the line $aX + bZ + c = 0$ as a line in the $y = 1$ plane then by our equivalence relation we may take the point $(\frac{x}{y}, 1, \frac{z}{y})$ as the representative for some point (x, y, z) in 3-space. We choose the point, $(\frac{x}{y}, 1, \frac{z}{y})$, as our representative because it is now a solution for the equation of our line. By performing a change of variables where we let $X = \frac{x}{y}$ and $Z = \frac{z}{y}$ we can rewrite the equation for our line as:

$$aX + bZ + c = a\frac{x}{y} + b\frac{z}{y} + c = 0$$

Upon multiplying through by y we get

$$ax + bz + cy = 0$$

We claim this line $ax + bz + cy = 0$ is now well defined in projective space. To see that it is well defined suppose we have (x_0, y_0, z_0) that satisfies $ax_0 + bz_0 + cy_0 = 0$ and we take the point $(2x_0, 2y_0, 2z_0)$, (equivalent to (x_0, y_0, z_0) by our equivalence relation \sim), and plug it into the equation of our line. We see now

$$a2x_0 + b2z_0 + c2y_0 = 2(ax_0 + bz_0 + cy_0) = 2 \cdot 0 = 0$$

and $(2x_0, 2y_0, 2z_0)$ is a solution to the equation of our line. Similarly any point in our equivalence relation also satisfies the equation, thus our equation is well defined in projective

space.

Now that we are able to define lines in projective space we may discuss parallel lines. We know in Euclidean space that two lines are parallel when their slopes are equal. That is $z = \frac{a}{b}x + c$ is parallel to $z = \frac{a}{b}x + d$, because the slopes of the lines, $\frac{a}{b}$, are equal. To simplify let us multiply through by b to get the equations $bz - ax - cb = 0$ and $bz - ax - db = 0$. Again we may use a change of variables to define these lines in projective space to get the projective lines $bz - ax - cby = 0$ and $bz - ax - dby = 0$. How do we expect these parallel lines to act in projective space? If we think about these lines as an artist would we would expect these lines to intersect. The linear perspective examples we have seen in previous sections suggest that parallel lines appear to converge to vanishing point. To find this point of intersection let us look at these lines in the $z = 1$ plane. By letting $z = 1$ we have equations in x and y : $b - ax - cby = 0$ and $b - ax - dby = 0$. If we solve these equations for x we get $x = \frac{b}{a} - \frac{cb}{a}y$ and $x = \frac{b}{a} - \frac{db}{a}y$. Clearly these two lines will intersect at the point $\frac{b}{a}$ when $y = 0$. Which is precisely what we should expect from our previous example where we found our points at infinity to be in the plane $y = 0$. This result of two parallel lines meetings at infinity is a defining property of projective space. Contrary to Euclidean space where parallel lines never intersect, in projective space *any* two lines will intersect.

From this artistic approach we have constructed modern projective geometry as a space. Beginning only with the notion of one point in projective space for each line through the origin in Euclidean 3-space, we have constructed a mathematical system that is consistent with the principles of linear perspective.

6 Conclusion

From the above examples it is clear there is a unique relationship shared between linear perspective in art and projective geometry in mathematics. The early understandings of projective geometry came directly out of the use of linear perspective in art and a curiosity

of how linear perspective works. Through its modern development projective geometry is now able to explain the principles of linear perspective as being ‘the correct’ thing to do when one wants to create a perspective image. Each field owes much to the other regarding their development both in the past and in the future. Swedish mathematician Gustaf Mittag-Leffler was right when he said “mathematical genius and artistic genius touch one another,” and beyond that they overlap and interact in a truly unique manner.

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