The Geometry and Statistical Analysis of Music

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1. Introduction

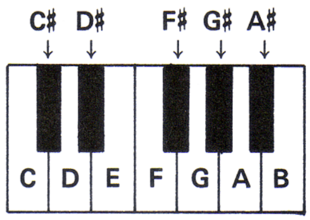
According to the book A Geometry of Music, by Dmitri Tymoczko, in Contemporary Music Theory, it is believed that musicians take on a mathematical approach when creating progressions of chords in their music. Because of this mathematical approach, musicians construct music as opposed to composing it [9] . In the construction of their music, musicians strive to create sound that is pleasing to their audience's ears. In order to reach this goal, musicians stray away from creating progressions of chords that are chaotic and random because these progressions create noise that is not pleasant to the ears. Musicians also tend to avoid progressions that are completely symmetrical because symmetrical progressions are boring and not appealing to the ear. When musicians construct their music, they strive to create progressions of chords that are almost-symmetrical [10]. Because of these almost symmetrical progressions of chords, barriers between musical styles have fallen and Western music of the twentieth century in fact is related to the classical music of past centuries. The music of these different eras are related because they contain progressions of chords that create similar geometric patterns that are short and efficient [9]. In this paper, we will first explore the math that underlines Contemporary Music Theory. We will then use the theory to analyze two songs from different eras.

2. Musical Theory

2.1. **Definitions.** A musical note in Western music is either a single sound or its representation in notation [3]. A pitch of a note is the frequency of its vibrations and can be represented by $p = c_1 + c_2 \; log(f/440)$, where c_1 is the number a note is assigned and c_2 represents the number of notes in an octave [9]. An octave is the twelve tone above a given pitch, with twice as many vibrations per second, or below a given pitch, with half as many vibrations. A piano has eight octaves and each octave has a different set of the following twelve notes C, C#/D♭, D, D#/E♭, E, F, F#/G♭, G, G#/A♭, A, A#/B♭, and B. All notes that are exactly an octave apart belong to the same pitch class. For example, all of the notes D# in the eight different octaves belong to the same pitch class. The smallest interval on a piano, a half step, or the distance between a note on a piano is known as a semitone [9].

We will represent a pitch's octave by a subscript. For example, the note C that lives in the first octave will be represented by C_1. However, in this paper, we will ignore octaves and assume that all notes in the same pitch class are the same note.

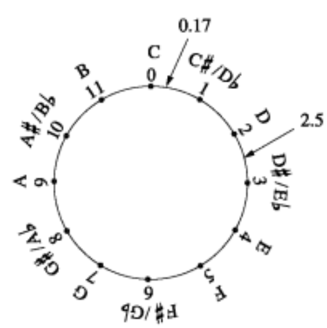
For example, since the notes $C_1, C_2, ..., C_8$ are in the same pitch class, we will assume that $C = C_1 = C_2 = ... = C_8$. Below is a picture of the twelve notes that appear in each of the eight octaves on a piano.



2.2. **Modular Arithmetic.** Each of our twelve notes will have a numerical representation between the range $0 \leq x \leq 11$:

C=0	**F#/G♭=6**
C#/D♭=1	**G=7**
D=2	**G#/A♭=8**
D#/E♭=3	**A=9**
E=4	**A#/B♭=10**
F=5	**B=11**

Because of this repetition of notes along our piano, we can view all of the notes on a piano on a one-dimensional circle that contains only twelve notes. Each time we make a full rotation around the circle, we enter the next identical octave on the piano. Below is the one-dimensional circular representation of our twelve notes:



[9]

Since we are dealing with this repetitive cycle around a one-dimensional circle and each of our twelve notes have numerical representation in the range $0 \leq x \leq 11$, we will use modular arithmetic $\mathbb{Z}_{12}$ to represent our notes and to express the progression of our notes. In $\mathbb{Z}_{12}$, $x \equiv y$ mod 12, if $12|(x-y)$ [11].Since we use modular arithmetic, we will be assuming that all notes in the same pitch class are congruent $\mathbb{Z}_{12}$. For example, if we look at the note D_1, its numerical representation is 2 mod12 and it can be represented on our one dimensional circle. If we begin at the note D_1 and move up by 12 semitones, or make a full rotation, we land on the D_2. Since we are working in $\mathbb{Z}_{12}$, D_2 can be represented numerically by $D_2 = 2 + 12 = 14 \equiv 2$ mod12 since $12|(14-2)$. Therefore, $D_1 \equiv D_2$ mod 12.

2.3. **Voice Leadings.** A chord is the simultaneous sounding of two or more notes [3]. A piece of music that includes two or more voices singing or two or more instruments playing notes simultaneously is an example of a piece of music containing chords. When a flute, violin, and two voices are playing and singing a note at the same time, a chord is being played. Chords will be represented in vector notion. For example, suppose we have a chord that contains the notes C and F. Since C=0 and F=5, our chord will be represent as $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$.

A voice leading is the the progression of one chord to another [9]. The displacement of a voice leading is the absolute value of the distance that each note moves in a progression [9]. We will be representing a displacement by a vector with curly brackets and the numbers in the displacement vector will be listed from greatest to least and a pre-order. In mathematics, pre-orders are binary relations that are reflexive and transitive. Consider some displacement P and a binary relation $\leq$ on P. Then $\leq$ is a pre-order if it is reflexive and transitive, i.e., for all integers a, b and c in P, we have that a $\leq$ a, reflexivity. Also if a $\leq$ b and b $\leq$ c then a $\leq$ c, transitivity [7]. There are numerous possible voice leadings that can occur between a progression of chords. The minimal bijective voice leading is the smallest of all the possible voice leadings [9]. Suppose there is a voice leading from $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ to $\begin{bmatrix} 11 \\ 8 \end{bmatrix}$. The minimal bijective voice leading is when the first note of the chord moves down by 1 semitone and the second note moves up by 3 semitones. The displacement of this voice leading is represented by the vector $\left\{ \begin{matrix} |-1| \\ |3| \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\}$. Another voice leading for this progression is when the first note moves up by 11 semitones and the second notes moves down by 9 semitones. The displacement of this voice leading is $\left\{ \begin{matrix} |11| \\ |-9| \end{matrix} \right\} = \left\{ \begin{matrix} 11 \\ 9 \end{matrix} \right\}$.

In order to study and analyze the progression of chords, we must view chords geometrically. In order to do this geometry, we must create a way to measure and compare the distance moved in a voice leading. There are numerous ways to measure a displacement vector of a voice leading. However, in this paper we will be using parsimony as a method of comparing voice leadings. Parsimony is related to the lexicogaphic ordering. It generates a notion introduced by Richard Cohn and developed by Jack Donthett and Peter Steinback. In parsimony, when given two voice leadings, α and β, α is smaller (or more parsimonious) than β if and only if there exist some real number d such that for all real numbers $c > d, c$ appears the same number of times in the displacement multiset of α and β, and d appears fewer times in the displacement mulltiset of α and β [10]. For example, according to Parsimony, the vector $\left\{\begin{matrix} 3+\epsilon \\ 0 \\ 0 \end{matrix}\right\}$ is larger than $\left\{\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}\right\}$ and $\left\{\begin{matrix} 4 \\ 3 \\ 3 \\ 3 \end{matrix}\right\}$ is larger than $\left\{\begin{matrix} 4 \\ 3 \\ 0 \\ 0 \end{matrix}\right\}$.

2.4. **Methods of Comparing Voice Leadings.** In order to create a geometry of chords that is almost-symmetrical and easy to understand, a method of comparing voice leading must satisfy the Distribution Constraint. Parsimony is a method of comparing voice leadings that satisfies the Distribution Constraint. A method of comparing voice leadings satisfies the Distribution Constraint when the following inequalities hold [9]:

$$\left\{\begin{matrix} x_1+c \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix}\right\} \geq \left\{\begin{matrix} x_1 \\ x_2+c \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix}\right\} \geq \left\{\begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix}\right\} \text{ for } x_1 > x_2, c > 0.$$

Below is an example of how parsimony satisfies this constraint:

Let $c = 1, x_1 = 5, x_2 = 3$, and $x_3 = 2$. When using parsimony, we know that the following inequalities hold, $\left\{\begin{matrix} 6 \\ 3 \\ 2 \end{matrix}\right\} \geq \left\{\begin{matrix} 5 \\ 4 \\ 2 \end{matrix}\right\} \geq \left\{\begin{matrix} 5 \\ 3 \\ 2 \end{matrix}\right\}$. This implies that $\left\{\begin{matrix} 5+1 \\ 3 \\ 2 \end{matrix}\right\} \geq \left\{\begin{matrix} 5 \\ 3+1 \\ 2 \end{matrix}\right\} \geq \left\{\begin{matrix} 5 \\ 3 \\ 2 \end{matrix}\right\}$.

A n-note multi-set of pitches is a chord that has n-notes and where there is a possibility of a repetition of the same notes [10]. Suppose we have any two n-note multi-set of pitches P and Q, where

$$P = \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ . \\ p_n \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_n \end{bmatrix}.$$

The voice leading from P to Q is crossing free, or uncrossed, if $p_i > p_j$ implies that $q_i \geq q_j$ for all i, j [10]. For example, the voice leading below is uncrossed since 6 > 5, 4 > 2, 5 > 3, and 2 > 1,

$$\begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, 6 > 5 \longrightarrow 4 > 2, 5 > 3 \longrightarrow 2 > 1$$

There is a strong relation between the avoidance of voice crossings in voice leadings and the geometry that creates symmetries that are easy to understand. If a method of comparing voice leadings obeys the distribution constraint, then there is a minimal voice leading between any two chords that is crossing free. Also, if a method of comparing voice leadings violates the distribution constraint, some crossed voice leading will be smaller than its natural uncrossed alternative. Theorem 1 is the result of these ideas and plays a major role in the theory of music.

Theorem 2.1. *Theorem 1 (Tymoczko)* [10]*: Let P and Q be any two n-note multi-set of pitches, and let our method of comparing voice leadings be a total pre-order satisfying the distribution constraint. Then there will exist a minimal bijective voice leading from P to Q that is crossing free. If the total pre-order strictly satisfies the distribution constraint, then every minimal bijective voice leading from P to Q will be crossing free.*

Below is a proof of part of Theorem 1. Through a proof by contradiction, we will prove that if we have method of measuring voice leadings that strictly satisfies the distribution constraint, then every minimal bijective voice leading from P to Q will be crossing free [10].

Proof. Suppose we fail to satisfy the first inequality of the displacement constraint,

$$\begin{bmatrix} x_1 + c \\ x_2 \end{bmatrix} < \begin{bmatrix} x_1 \\ x_2 + c \end{bmatrix} \text{ with } x_1 > x_2 \text{ and } c > 0.$$

Let A be the *crossed* voice leading, $\begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + c \\ x_1 \end{bmatrix}$.

$\implies$ displacement of $A = \begin{Bmatrix} x_1 + c \\ x_2 \end{Bmatrix}$.

Let B be the *uncrossed* voice leading, $\begin{bmatrix} 0 \\ x_1 - x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_1 + c \end{bmatrix}$.

$\implies$ displacement of $B = \begin{Bmatrix} x_1 \\ x_2 + c \end{Bmatrix}$.

This implies that the displacement of A is smaller than the displacement of B. Therefore we have shown that a crossed voice leading is smaller than an uncrossed voice leading.

Suppose we fail to satisfy the second inequality of the displacement constraint and $\begin{bmatrix} x_1 + c \\ x_2 \\ x_3 \end{bmatrix} < \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x_3 > x_2 > 0$ and $c > 0$.

Let A be the *crossed* voice leading, $\begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 + c \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_1 + c \end{bmatrix}$.

$\implies$ displacement of $A = \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix}$.

Suppose we add pitch $p = 0$ to the first chord and map it to some note in the second chord.

Let B be the *crossed* voice leading, $\begin{bmatrix} 0 \\ x_1 + x_2 \\ x_1 + x_3 + c \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + c \\ x_1 \\ x_1 + c \end{bmatrix}$.

$\implies$ displacement of $B = \begin{Bmatrix} x_1 + c \\ x_2 \\ x_3 \end{Bmatrix}$.

Let C be the *uncrossed* voice leading, $\begin{bmatrix} 0 \\ x_1 + x_2 \\ x_1 + x_3 + c \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_1 \\ x_1 + c \end{bmatrix}$.

$\implies$ displacement of $C = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$.

This implies that the displacement of C is greater than the displacement of A.

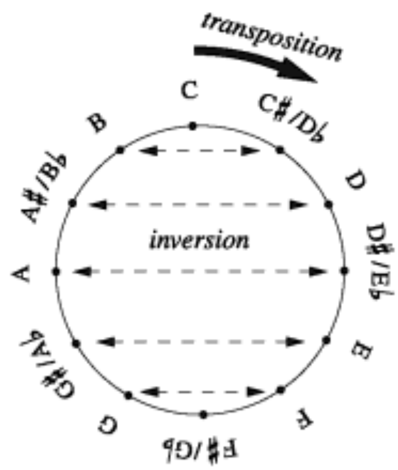
Therefore we have an uncrossed voice leading that is larger than a crossed voice leading. In conclusion, this proof shows that if we have a method of comparing voice leadings that does satisfies the Distribution Constraint, then the minimal bijective voice leading between two chords is uncrossed. $\square$

2.5. **Distance Preserving Functions.** The idea of distance-preserving transformations of musical space play a large role in composing music because musicians are primarily sensitive to the distance between notes. Transposition and inversion are the only two types of distance-preserving transformations that exist. These two distance-preserving transformation not only play an important role in many different musical styles, but are key to Contemporary Music Theory.

Transposition is a type of distance-preserving function that moves every pitch in a chord the same distance in the same direction [9]. Transposition of pitch p by x semitones is represented as $T_x(p) = p + x$. Suppose we have a chord $x = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ and we want to translate our chord down by 5 semitones. This transposition function will look like $T_{-5}(x) = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \rightarrow \begin{Bmatrix} -5 \\ -5 \end{Bmatrix} \rightarrow \begin{bmatrix} 7 \\ 0 \end{bmatrix}$, where $\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$ is the displacement vector of the transposition.

Inversion is a type of distance-preserving function that turns the musical space upside down. In this function, the direction of motion changes, while the distance between each pitch remains the same. Inversion can be represented mathematically by subtraction from a constant value [9]. The inversion that maps p to y, where p and y are pitches, is represented as $I_x(p) = x - p = y$. Suppose we have a chord $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose we want to invert our chord up by 4 semitones. This inversion will look like $I_4(p) = \begin{bmatrix} 4-1 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, where $\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ is the displacement vector of the transposition. Figure 2.34 shows how transportation and inversion can be defined in pitch-class space.

Figure 2.3.4 In circular pitch class space, transposition corresponds to rotation, while inversion corresponds to reflection.



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Another type of function that plays an important role in Contemporary Music Theory that is not a distance-preserving transformation is the permutation function. The permutation function is represented by $\sigma(x)$ and simply permutes the pitches in a multi-set of pitches. For example, suppose we have the multi-set of pitches P, where $P = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Then a permutation of P would be $\sigma(P) = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}$. In other words, the permutation function rearranges the order of the pitches in a multi-set of pitches.

A property of a multi-set of pitches that plays an important role in creating efficient voice leadings, chord structures and symmetry is whether or not a multi-set is invariant under a function. A multi-set of pitches x is said to be invariant under a function G if $G(x) = \sigma$(x)$\approx$ x [9]. Suppose our function is a transposition of +4, G $= T_4(x)$. A multi-set that is invariant under G is $\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$, since $\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} \rightarrow \begin{Bmatrix} 4 \\ 4 \\ 4 \end{Bmatrix} \rightarrow \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix} =$ $\sigma x \approx x$.

The existence of a multi-set that invariant under a a function creates important theorem that is key in creating efficient voice leadings, chord structures, and almost-symmetry.

Theorem 2.2. *Theorem 2 (Tymoczko): Let A be a n-note multi-set of pitches and let $G(x)$ be some distant preserving function. If there exists a S_G that is an invariant* *symmetry under a function, the distance between A and S_G creates some upper bound for the distance between A and G(A), where this distance is related to the distance between A and S_G . This upper bound is 2 times the displacement vector from A to S_G. Also, the distance between A and G(A) creates some upper bound for the distance between A and S_G.*

Below is an example of a chord progression that satisfies Theorem 2:

Suppose we have a transposition function $G(x)$=$T_4(x)$, that translates all notes in a chord up by 4 semitones. A chord that is invariant under this function is S_G=$\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$ since $\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} \rightarrow \begin{Bmatrix} 4 \\ 4 \\ 4 \end{Bmatrix} \rightarrow \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix} = \sigma x \approx$ x. Let A=$\begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$.

When we plug our chord A into our function $G(x)$, we get

$G(A) = T_4(\begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix})$ $\begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix} \rightarrow \begin{Bmatrix} +4 \\ +4 \\ +4 \end{Bmatrix} \rightarrow \begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix}$.

If we ignore order, then $\begin{bmatrix} 4 \\ 8 \\ 11 \end{bmatrix} \approx \begin{bmatrix} 11 \\ 4 \\ 8 \end{bmatrix}$ and we can see that when we plug A into our function G(x), the distance that A moved can be represented by the displacement vector $\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$.

The distance between A and S_G is $\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ since A = $\begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix} \rightarrow \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \rightarrow \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} =$ S_G. This implies that the chord A is almost-symmetrical since it is only one semitone away from our invariant function S_G,

The distance between S_G and $G(A)$ is $\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ and this progression can be seen below,

S_G=$\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} \approx \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix} \rightarrow \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \rightarrow \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = G(A)$

We now can look at the distance between A and $G(A)$ in two ways,

(1) A$\rightarrow$ $G(A)$.

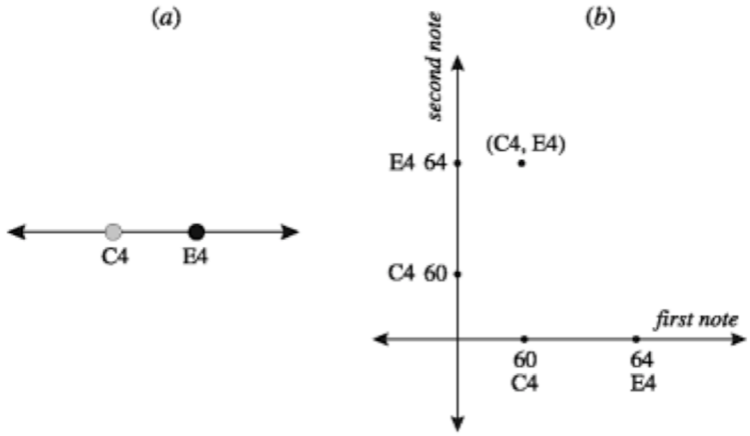
The amount moved or displacement vector of this voice leading is $\left\{ \begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix} \right\}$, which means that the notes move by 2 semitones.

(2) A $\rightarrow S_G \rightarrow G(A)$ The amount moved or the displacemtn vector of this voice leading is $\left\{ \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 0 \\ 0 \end{smallmatrix} \right\}$, which means that the notes move by 2 semitones.

If we look at 2 times the distance between A and S_G, we get $2 \times \left\{ \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 0 \\ 0 \end{smallmatrix} \right\}$ and the notes move by 2 semitones. Also, since the notes move by 2 semitones in the voice leading from A to G(A), we can see that the distance between A and S_G creates an upper bound for the distance between A and G(A). This upper bound is 2 times the distance between A and S_G and the Distribution Constraint ensures that $\left\{ \begin{smallmatrix} 2 \\ 0 \\ 0 \end{smallmatrix} \right\}$ $> \left\{ \begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix} \right\}$.

3. The Geometry of Chords

3.1. **Ordered Pitch Space and Mӧbius Geometry.** The relationship between music and geometry begins with chords being represented as points in higher dimensional spaces. There are two ways to represent ordered sequences of pitches geometrically. The ordered sequences can be represented in a one-dimensional space, figure (a), where progressions between chords are represented by collections of paths. In this representation, we can ignore the octave of each pitch and assume that all pitches in the same pitch class are the same, e.g.$(0 = C_1 = C_2 = ... = C_8)$. By ignoring octaves, we can wrap the line into a circle. This prevents the reproduction of identical pitch class spaces.



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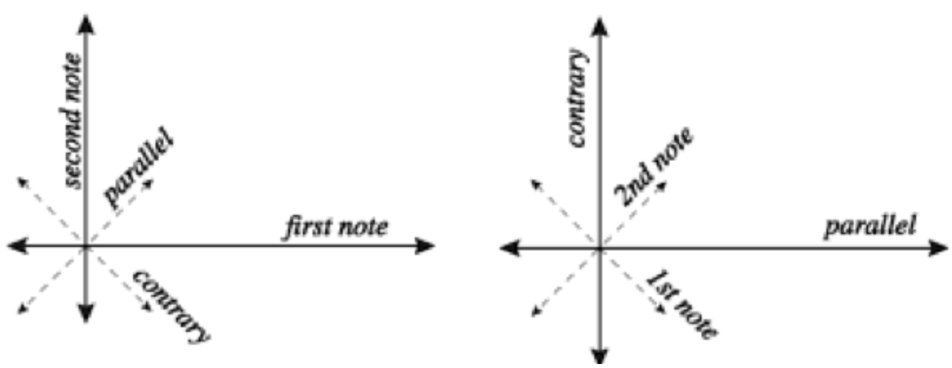
The second representation is a two-dimensional space, figure (b), where the horizontal axis represents the first note of the chord, while the vertical axis represents the second note. Unlike the one-dimensional space, in this space it is difficult to ignore the octaves and order of each note and progression is represented by line segments instead of a collection of paths. In the two-dimensional space, the horizontal and vertical line segments represent motion in a single voice in a chord. Diagonal motion with positive slope represents parallel motion and signifies that the two notes in the chords are translating by the same amount, in the same direction. Where as diagonal motion with negative slope represents contrary motion and this signifies that the two voices in the chord translate by the same distance but in different directions.



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The figure above shows two representations of the different motions in our two-dimensional space. However, the second representation is what the motion looks like after a a clockwise rotation of 45 degrees takes place. The result of this rotation is that all four of the the different movements change axes. This rotation is necesary because it will allow parallel motion to be represented on the horizontal axis, while the perfect contrary is represented on the vertical axis. These new movements allow the chords on the same vertical line to sum to the same value and chords on the same horizontal line to relate by transposition. Although the axes change roles in this rotation, the space does not change [9].

Figure 1.1 in the appendix depicts an extended potion of a two-dimensional ordered pitch space with the axis rotated by 45 degrees. When viewing this pitch space, one can see that periodicity occurs. The larger plane consists of 4 single patterned planes. Each of the 4 planes, or tiles, are related in that they all contain the same chords [9]. However, the chords in each tile differ from the chords in the other tiles by the octave and ordering of the notes.

The chords in the lower left tile are related to the upper right tile by octave transposition. In both of these tiles, the first note in each chords is the same. While the second note in each of the chords in the lower left is one octave below the second note of the chords in the upper right tile. Similar patterns occur between the upper left and lower right tiles. The first note in the chords in the upper left tile are one octave below than the first note in the chords in the lower right tiles. The rest of the infinite space can be created since diagonal motion between tiles always correspond to octave transposition.

The relationship between the lower and upper left tiles is hard to visually grasp. In order to see this relationship, imagine that there were hinges connecting the upper left and lower left quadrants so that the bottom left could be flipped onto the top left. This transformation causes each chord to match up with a chord that has the same notes, but just in reverse order. This transformation is a reflection, meaning a chord from the lower left tile maps to a chord with the same notes, just switched order. The same relation applies to the upper and lower right tiles. The figure below represents these relationships, except the ordering of chords from top to bottom is ignored.

[9]

If we view identify the two-dimensional pitch space as tiles, then the space we are dealing with is a plane that is much more interesting than any ordinary plane. We will now see that this plane contains twists, mirrors, and m$\ddot{o}$bius strips [9]. In order to show these traits, we will began with the "Parable of the Ant" [9]. Suppose an ant is walking along figure 3.2.1(a) and we want to bet on whether it will touch a pipe when walking across the figure. We will assume that the pipe that the ant touches does not matter, just as long as it touches one of the four pipes. Since the figure is just a wallpaper of the same face, four times, we can just represent the ant's trajectory as a single tile, figure 3.2.1(b).

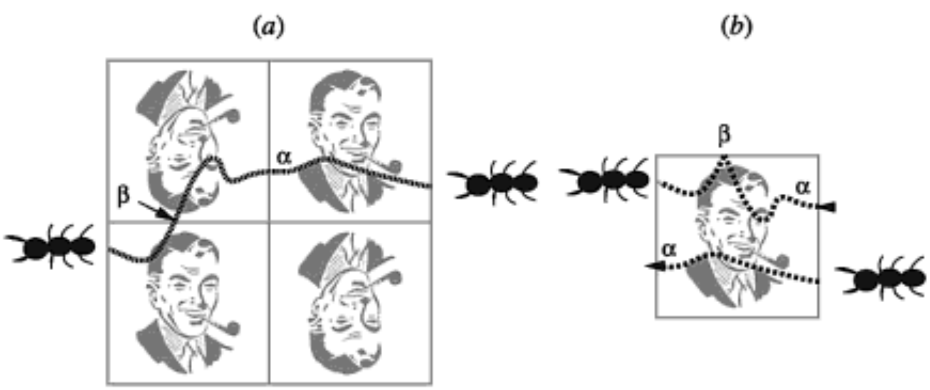
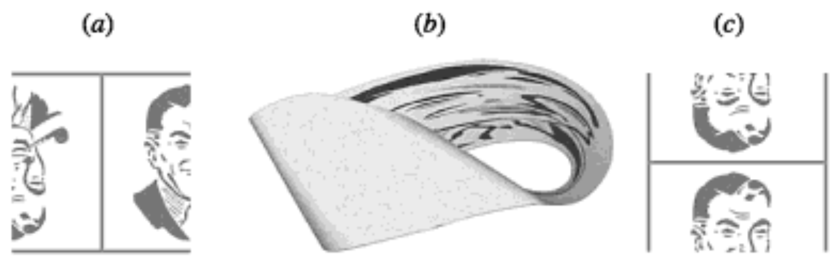


Figure 3.2.1
An ant's path car be represented on a single tile.

[9]

On figure 3.2.1(b), at the point marked α, the ant disappears at the lower left edge of the the upper right tile. In part (a), we can see it just goes onto the upper left tile. However, if we look at just the one tile, it reappears at the upper right edge. At the point β, we can see that on (a), it goes off of the bottom edge of the upper left tile and it enters on the top edge of the lower left tile. However, we can see on the single tile that at point β it bounces off of the top of the tile. This shows that the ant's trajectory can be displayed on a single tile. However, the top and bottom act as bumpers and the left and right sides are reminiscent of early video games such as asteroids or Pac-Man. There is a slight difference between the movement of the Pac-Man and the ant. When Pac-Man goes off a side he reappears on the opposite side directly across from where he left the other side. The ant, however, does not reappear directly across, he reappears somewhat diagonal. The idea of Mӧbius Geometry comes into play because of this movement. Suppose we display part of figure 3.2.1(a) into a figure without any left or right boundaries, figure 3.2.2(a). If we attach the two edges, we would get a Mӧbius Strip, 3.2.2(b), since after the twist the edges would match up with itself.

Figure 3.2.2 (*a*) The choice of left and right boundaries is arbitrary. (*b*) We could even represent the wallpaper without any left and right boundaries at all, if we stretched it horizontally and used the third dimension to attach the two edges. (*c*) The choice of upper and lower boundaries is not arbitrary, since this figure has no pipe.



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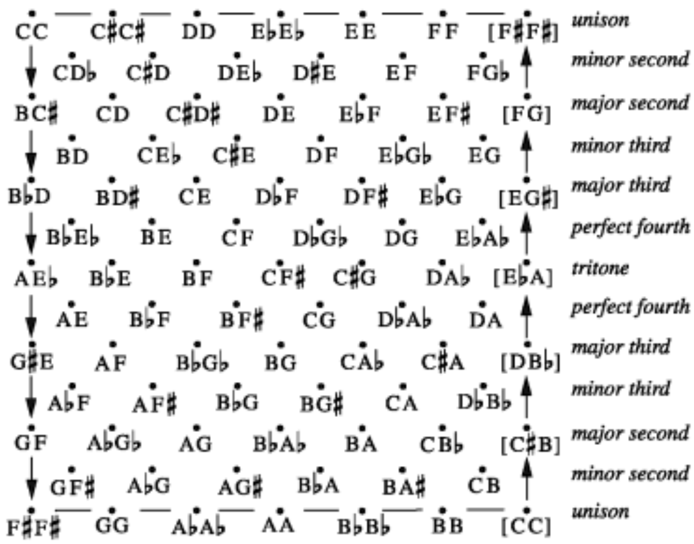
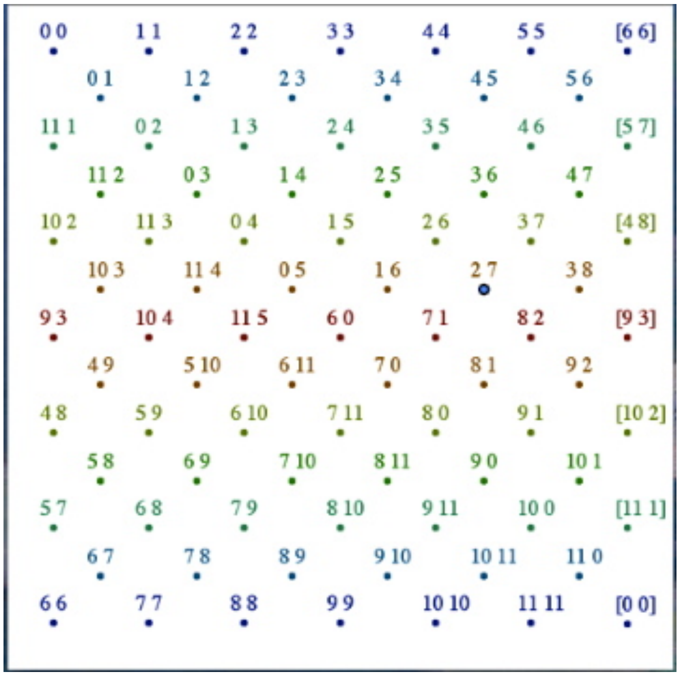
The "Parable of the Ant" has a key idea that we will now relate to the two dimensional musical space. We will now assume that the octave and ordering of each pitch in the chords does not matter, e.g.($0 = C_1 = C_2 = ... = C_8$ and (C,D)=(D,C)). Now each of the four tiles in figure 1.1 in the appendix contains precisely one point from every unordered set of pitch classes. Because of this, we can use one tile to represent our two-dimensional ordered pitch space. Suppose we choose the upper right tile as our representation of the musical space. Just like figure 3.2.2(b) from the "Parable of the Ant", our new figure, figure 3.3.1, has the characteristics of the top and bottom acting as bumpers and the right and left acting as video games.

Figure 3.3.1 Two-note chord space. The left edge is "glued" to the right, with a twist.



[9]

Our goal is to twist figure 3.3.1, just as we did with the figure from the "Parable of the ant". This will work because once we attach the edges by a twist, each of the chords (0,0),(11,1),(9,3),(8,4),(7,5) and (5,6) will meet up with its duplicate. In our new fold, we again rotated the axes so that parallel musical motion is represented by horizontal geometric motion, while vertical motion represents contrary motion. Since we have duplicate chords that appear on opposite edges, this geometric space depicts a Mӧbius Strip [9]. The image below is the numerical representation of figure 3.3.1. We will be using this to represent the progressions of chords.



[9]

Figure 1.2 in the appendix is an example of a chord progression that is represented on the numerical Möbius Strip.

4. Statistical Analysis

4.1. **Introduction.** In the first part of this paper, we discussed how in Contemporary Music Theory it is believed that composers of music take on a mathematical approach when creating music and construct music as a opposed to composing music. Because of these underlying geometric constructions, many musical theorist today believe that Western music of the twentieth century is in fact related to the classical music of the past centuries [9]. In the second part of this paper, I will compare and contrast the statistical analysis of two-dimensional chords from two different pieces of music that were composed in different eras. The two songs that I will be comparing are Johann Pachelbel's most famous classical piece of music, “Canon in D”, and the popular country pop song, “Need You Now”, by Lady Antebellum.

4.2. **Lady Antebellum's “Need You Now”.** In 2006, an American country pop group by the name of Lady Antebellum was formed in Nashville,Tennessee. The group released their second number one single “Need You Know” in August 2009. The song “Need You know” was co written by Lady Antebellum and Josh Kear. It received four Grammy Awards in 2011, including Song of the Year and Record of the Year

[5]. "Need You Know" is a song that is composed of numerous progressions of chords that include harmonizations of voices and instruments. Because of the appearance of these types of chords in this song, I will explain the statistical analysis of this song.

4.3. **Statistical Analysis of "Need You Now".** The chords that I will be analyzing in the music of "Need You Now" are two-dimensional chords that include two different voices singing simultaneously. In the music of this song, there are about 9 stanzas that include progressions of these two dimensional chords [4]. The 9 different stanzas include interesting patterns, almost-symmetries, and voice leadings that help prove that the composers of this song strived for short and efficient voice leadings when composing the music of this song.

Since we are assuming that all octaves are the same, we are working with only 12 different notes. Since we are working with 12 notes and two-note chords, there are 144 possible combinations of chords. However, since we are assuming that the ordering of the notes in the chord does not matter and since this song is written in E major, there are only 49 possible combinations of these two-note chords [4]. From the statistical analysis of these two-dimensional chords in the song "Need You Now", I found that there are only 11 different chords played throughout the song. This shows that in this piece of music, only about 22.5% of all the possible two-dimensional voice chords actually appear. The 11 chords that are present in "Need You Now" are (G#,E) $= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$., (F#,D#) $= \begin{bmatrix} 6 \\ 3 \end{bmatrix}$., (B,G#) $= \begin{bmatrix} 11 \\ 8 \end{bmatrix}$., (E,C#) $= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$., (D#,B) $= \begin{bmatrix} 3 \\ 11 \end{bmatrix}$., (C#,A) $= \begin{bmatrix} 1 \\ 9 \end{bmatrix}$., (B,E) $= \begin{bmatrix} 11 \\ 4 \end{bmatrix}$., (A,F#) $= \begin{bmatrix} 9 \\ 6 \end{bmatrix}$., (F#,C#) $= \begin{bmatrix} 6 \\ 1 \end{bmatrix}$., (B,B) $= \begin{bmatrix} 11 \\ 11 \end{bmatrix}$., (C#,G#) $= \begin{bmatrix} 1 \\ 8 \end{bmatrix}$. Each of the 11 chords appear in this song a different amount of times. Figure 2.1 in the appendix is a table that shows which chords appear in this piece of music, the amount of times they are played, and the density of the appearance of each of the chords.

Figure 2.2 in the appendix is a picture of the two-dimensional Möbius Strip that represents the pitch space. The picture shows the density of each of the chords in this piece of music. The density of each chord is represented by a red circle. The less transparent the chord is, the more it occurs in the piece of music.

I began noticing patterns and almost-symmetry in "Need You Now" when I began the statistical analysis of the voice leadings. Figure 2.3 in the appendix is a chart that displays the voice leadings of the two dimensional voice chords that occur in this song and the amount of times that they occur. From figure 2.1 in the appendix, we can see that the two-dimensional voice chord that is played the most in this piece of music is

(G#,E) $= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, it is played 39 times [4]. When we begin to look at the voice leadings that begin at this chord, it is very common that this chord progresses to itself. In fact, out of the 39 voice leadings that begin at $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$, 18 of these progressions make no movement and just stay at $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$. The displacement of this voice leading is $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. The next most common voice leading for this chord is $\begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 3 \end{bmatrix} =$ (F#,D#). The displacement of this voice leading is $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$.

The voice leading from $\begin{bmatrix} 8 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ is not interesting and creates no geometry for us to analyze. However, the voice leading from $\begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 3 \end{bmatrix} =$ (F#,D#) is interesting because it is an example of a voice leading that is almost-symmetric. Since this voice leading has a high density, I decided to analyze it more. I went back to the music of this song and found that it was very common that when this voice leading occurred there was another voice harmonzing with these notes. The voice leading that was present was $\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 11 \\ 6 \\ 3 \end{bmatrix}$ and I discovered that this was in fact a popular progression of multi-sets that occurs in this song. The first multi-set contains the chord $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and the second contains the chord $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$. The reason why this voice leading is so common is because each of these chords are close to vectors that are symmetric and create smooth transitions between chords. Below shows the progression of these multi-sets that occur in this song,

$$\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} \rightarrow \begin{Bmatrix} 0 \\ 0 \\ |-1| \end{Bmatrix} \rightarrow \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} \rightarrow \begin{Bmatrix} |-1| \\ |-1| \\ |-1| \end{Bmatrix} \rightarrow \begin{bmatrix} 7 \\ 3 \\ 11 \end{bmatrix} \rightarrow \begin{Bmatrix} |-1| \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} 6 \\ 3 \\ 11 \end{bmatrix}.$$

In this progression, we can see that both of our multi-sets of pitches are close to multi-sets that are symmetrical. The multi-sets of pitches $\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 3 \\ 11 \end{bmatrix}$ are symmetric because if you add four semitones to each of the notes in both of the sets, it does not change either of the multi-sets of pitches. The fact that both $\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 3 \\ 11 \end{bmatrix}$ are right next to these symmetric multi-set of pitches shows that the composers of this song strived for almost-symmetry.

4.4. **Voice Leadings Represented on the Mӧbius Strip:** .

The follow chord progression, $\begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 \\ 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 \\ 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ is the chord progression that occurs in the 8th stanza of this song [4]. In 10 of these voice leadings, the displacement is $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$ because the same chord is being sung twice or three times in a row. These kinds of voice leadings are not interesting, so we will look at only the voice leadings in this stanza that progress to a different chord. The following chord progression $\begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 \\ 6 \end{bmatrix} \Longrightarrow \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ is the progression of chords that occurs in stanza 8 that does not include the voice leadings that are just a progression of a chord to itself. Figures 2.4(a),(b), and (c) are the geometric representation of these voice leadings on the two-dimensional Mӧbius Strip. The 9 different tiles represents each of the 9 different voice leadings that occur in this stanza.

From figures 2.4(a),(b), and (c), you can notice that the composers strived for almost-symmetry because the pictures of the voice leadings are neither chaotic nor perfectly symmetric. Also, from this figure, we can begin to see the statistics of this song. First, these pictures show us the common chords that are used. Just like the image of the Mӧbius Strip with red circles above, we too can determine the density of each of the notes. These representations of the voice leadings between chords allow us to visually see how much a chord moves in the progression instead of just relying on a displacement vector. The idea that musicians strive to construct voice leadings that are efficient is very apparent here because most of these voice leadings are short.

4.5. **Pachelbel's Canon.** Pachelbel's Canon was composed sometime in the pre-1700 century era and is known as the most famous piece of music by the German Baroque composer Johann Pachelbel. Since it was very common for the work of pre-1700 composers to be forgotten for centuries, Pachelbel's Canon remained unknown for many centuries. However, it was rediscovered in the 20th-century. The piece became extremely popular shorty after it was published in 1919. Pachelbel's Canon is still frequently played today at weddings and is included on classical music compilations. The Canon was originally scored for three violins and basso continuo and paired with a gigue in the same key [9]. For this analysis, we will study Pachelbel's

"Canon in D", which is also known as the a version of *Pachelbel's Canon* that has variations on a ground bass. It includes a flute and two pianos [6].

4.6. **Statistical Analysis of "Canon in D".** The chords that I will be analyzing in the music of "Canon in D" are two-dimensional chords that include two different instruments, a flute and a piano, playing simultaneously. I will be analyzing the whole song since at all times in this song, these two-dimensional chords are being played [6]. Like the two-dimensional voice chords in "Need You Now, these two-dimensional instrument chords in the "Canon in D" have almost-symmetries and voice leadings that help prove that the composer of this song strived for short and efficient voice leadings in their construction.

Since we are ignoring ordering of chords, we will be working with a possibility of 72 chords. However, since this song is written in D major, there are only 49 chords that could occur in this song [6]. From the statistical analysis of these two-dimensional chords in the song Canon in D, I found that there are only 25 different chords played throughout the song. This shows that in this piece of music, only about 51% of all the possible two-dimensional instrument chords actually appear. The 25 chords that are present in "Canon in D" are (B,G) = $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$., (F#,D#) = $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$., (C#,A) = $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$., (A,F#) = $\begin{bmatrix} 9 \\ 6 \end{bmatrix}$., (A,D) = $\begin{bmatrix} 9 \\ 2 \end{bmatrix}$., (D,D) = $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$., (A,A) = $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$., (D,B) = $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$., (D,G) = $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$., (E,A) = $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$., (F#,B) = $\begin{bmatrix} 6 \\ 11 \end{bmatrix}$. (A,G) = $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$., (E,D) = $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$., (G,F#) = $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$., (E,G) = $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$., (C#,F#) = $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$., (C#,G) = $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$., (F#,F#) = $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$., (B,A) = $\begin{bmatrix} 11 \\ 9 \end{bmatrix}$., (C#,D) = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$., (B,E) = $\begin{bmatrix} 11 \\ 4 \end{bmatrix}$., (G,G) = $\begin{bmatrix} 7 \\ 7 \end{bmatrix}$. (F#,E) = $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$., (B,B) = $\begin{bmatrix} 11 \\ 11 \end{bmatrix}$., (C#,E) = $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Each of the 25 chords appear in this song a different amount of times. Figure 2.5 in the appendix is a table that shows which chords appear in this piece of music, the amount of times they are played, and the the density of the appearance of each of the chords. Figure 2.6 in the appendix is a picture of the two-dimensional Mӧbius Strip that represents the pitch space. The picture shows the density of each of the chords in this piece of music.

Figure 2.7 in the appendix is a chart that displays the voice leadings of the two-dimensional instrument chords that occur in this song and the amount of times that they occur. After analyzing the different voice leadings, I noticed that in this piece of music the displacement vectors that occurred between the progressions of these chords were small and efficient. Below is a piece of this music that represents these efficient voice leadings.

4.7. **Voice Leadings Represented on the Möbius Strip:** The follow chord progression, $\begin{bmatrix} 4 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 7 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 4 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 7 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 9 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 11 \\ 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ is the chord progression that occurs in the 7th stanza of this song [6]. Unlike a majority of the voice leadings in "Need You Now", none of these voice leadings has the displacement $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$. This is a progression that does not include the voice leadings that are just a progression of a chord to itself. An interesting pattern that occurs in this voice leading is that the displacement of these voice leadings moves one of notes in the chord by only one or two semitones. This is very small movement and shows in the construction of this piece of music, Pachelbel strived for voice leadings that were efficient and short. Figures 2.8(a),(b),(c), and (d) are the geometric representation of these voice leadings on the two-dimensional Möbius Strip. The 14 different tiles represents each of the 14 different voice leadings that occur in this stanza.

From figures 2.8(a),(b), (c) and (d), we not only can see the small displacement vectors, but we can notice that the composers strived for almost-symmetry because the pictures of the voice leadings are neither chaotic nor perfectly symmetric. The voice leadings in this song are much closer to being symmetric than the voice leadings from "Need You Now". Again, from this figure, we can begin to see the statistics of this songs.

We can began to compare figures 2.8(a),(b), (c) to figures 2.4(a),(b) and (c). There are many similarities that occur between these two songs. In both figures, the action that takes place in these voice leadings occur in the same areas of the Möbius Strip. In both of the representations of the chord progression, the voice leadings usually occur in the bottom left and upper right of the Möbius Strip. Another important similarity is that both of these images represent that the composers of both of the songs strived for an almost-symmetrical pattern and efficient voice leadings.

5. Conclusion

In this paper we have discussed how in Contemporary Music Theory it is believed that musicians take on a mathematical approach when creating the progressions of chords. After researching the relation between music and mathematics, I have learned that the underlying geometry of chords progressions helps theorist find similar geometric patterns and symmetries that occur throughout all types of genres of music. Being able to apply mathematics to music has allowed me to understand why musicians strive for certain constructions of progressions of chords. Lastly, through this research I have learned that mathematics is an essential tool in creating the art of music.

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